FREE VIBRATIONS OF SIMPLY SUPPORTED BEAMS USING FOURIER SERIES

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Abstract

Fourier series will be utilized for the solution of simply supported beams with different loadings in order to arrive at a free vibration. The equation of the free vibration is $\frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} = 0$

One of the methods of solving this type of equation is the separation of the variables which assumes that the solution is the product of two functions, one defines the deflection shape and the other defines the amplitude of vibration with time.

Modes of deflection with and without time along the beam were drawn for certain cases.

Good agreement has been obtained between the results of the present study and that determined by Timoshinko[11].

Key words: Beams, Fourier series, Free vibration, Structural analysis.

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$$\frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} = 0$$

[illegible]

.[11] Timoshinko [illegible]

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Notation:

EI	Flexural rigidity of the section of the beam.
L	Length of beam.
w(x)	Load function.
$\phi(x)$	Function of distance.
Y(t)	Amplitude of vibration with time.
t	Time
x	Distance
m	Mass
c	Constant
ω	Natural frequency of the beam.
α	$\sqrt{\omega / c}$

Introduction:

The study of large amplitude of simply supported beams can be traced to the work of Kreiger[1] wherein the governing partial differential equations were reduced to ordinary differential equations, and the solution was obtained in terms of elliptic functions using a one-term approximation. Similarly, Butgreen[2] gave the solution for the large amplitude vibration problems of hinged beams based on the classical continuum approach. Srinivasan employed the Ritz-Galerkin technique to solve the governing nonlinear differential equation of dynamic equilibrium for free and forced vibration of simply supported beams and plates [3, 4]. Evenesen[5] extended the study for various boundary conditions using the perturbation method.

Ray & Bert[6] carried out experimental studies to verify the analytical solutions for the nonlinear vibrations of simply supported beams and compared the solution schemes such as the assumed space mode, assumed time mode and Ritz-Galerkin methods and concluded that the latter two are better than the former. Pandalai & Sathyamoorthy [7] developed model equations for the nonlinear vibrations of beams, plates, rings and shells using Lagranges equations and highlighted the difference in the nature of the model equations for beams and plates, rings and shells.

Lou & Sikarskie[8] employed form-function approximations to study the nonlinear forced vibrations of buckled beams. Rehfield [9] used an approximate method of nonlinear vibration problems with material nonlinear effects for various boundary conditions.

Mustafa [10] used Laplace transformation method to solve the free vibration of simply supported beams.

Theory and Application

The partial differential equation (p.d.e) for free undamped transverse vibration of beams is [11]:

$$\frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} = 0 \qquad (1)$$

where $c^2 = \dfrac{EI}{m}$

One method of solving this equation is by the separation of variables;
it assumes that:

$$y(x,t) = \phi(x) Y(t) \qquad (2)$$

where ϕ(x) is a function of distance along the beam defining its deflection shape when it vibrates and Y(t) defines the amplitude of vibration with time.

Substituting equation (2) for equation (1) yields:

$$\phi(x) \frac{\partial^2 y}{\partial t^2} + c^2 Y(t) \frac{\partial^4 \phi(x)}{\partial x^4} = 0 \qquad (3)$$

The equation (3) is rewritten so that the variables x and t are collected together into separate terms as follows:

$$\frac{c^2}{\phi(x)} \frac{\partial^4 \phi(x)}{\partial x^4} = -\frac{1}{Y(t)} \frac{\partial^2 Y(t)}{\partial t^2} \qquad (4)$$

Since each of the variables x and t are independent variables, then each side of equation (4) is equal to a constant, say ω^2

It may be rewritten down to two ordinary differential equations that have to be satisfied:

$$\frac{c^2}{\phi(x)} \frac{\partial^4 \phi(x)}{\partial x^4} = \omega^2 \qquad (5)$$

Rearranging equation (5) yields:

$$\frac{\partial^4 \phi(x)}{\partial x^4} - \frac{\omega^2}{c^2} \phi(x) = 0 \qquad (6)$$

Putting $\alpha^4 = \dfrac{\omega^2}{c^2}$ yields: (7)

$$\frac{\partial^4 \phi(x)}{\partial x^4} - \alpha^4 \phi(x) = 0 \tag{8}$$

Equation (8) can be rewritten as:

$(D^4 - \alpha^4)\phi(x) = 0$ where $D = \dfrac{\partial}{\partial x}$

The auxiliary equation is:

$$D^4 - \alpha^4 = 0 \tag{9}$$

Analysing equation (9) yields:

$$(D^2 - \alpha^2)(D^2 + \alpha^2) = 0$$

then

$$D^2 = \alpha^2 \qquad\qquad D^2 = -\alpha^2$$

and

$$D = \pm\alpha \qquad\qquad D = \pm i\alpha$$

The general solution is given by:

$$\phi(x) = c_1 \sin\alpha x + c_2 \cos\alpha x + c_3 \sinh\alpha x + c_4 \cosh\alpha x \tag{10}$$

where c_1, c_2, c_3 and c_4 are constants

and

$$-\frac{1}{Y}\frac{\partial^2 Y(t)}{\partial t} = \omega^2 \tag{11}$$

Rearranging equation (11) yields:

$$\frac{\partial^2 Y(t)}{\partial t^2} + \omega^2 Y(t) = 0 \tag{12}$$

And rewriting (12) as:

$$(D'^2 + \omega^2)Y(t) = 0$$

where $D^{'} = \frac{\partial}{\partial t}$

Or

$D^{'2} + \omega^2 = 0$

and

$D^{'2} = -\omega^2$

Then

$D^{'} = \pm i\omega$

Then the general solution is given by:

$$Y(t) = A\cos\omega t + B\sin\omega t \qquad (13)$$

Substituting equations (10) and (13) for equation (2) yields:

$$y(x,t) = (A\cos\omega t + B\sin\omega t)(c_1\sin\alpha x + c_2\cos\alpha x + c_3\sinh\alpha x + c_4\cosh\alpha x) \qquad (14)$$

The complete solution for a particular structure requires expressions for the displacement, slope, moment and shear at the supports which must be substituted for (14). This procedure will yield three coefficients in terms of the forth and will also yield a frequency equation from which ω may be evaluated. The final coefficient expression is a magnitude of vibration that would require acknowledging of the initial conditions of motions.

For the simply supported beams the boundary conditions are:

$$y(0,t) = 0 \quad \text{and} \quad EI\frac{\partial^2 y}{\partial x^2}(0,t) = 0 \qquad (15a)$$

$$y(L,t) = 0 \quad \text{and} \quad EI\frac{\partial^2 y}{\partial x^2}(L,t) = 0 \qquad (15b)$$

Substituting equation (a,b) for equation (14) yields:

$$0 = c_1\sin\alpha L + c_3\sinh\alpha L$$

And

$$0 = -c_1\sin\alpha L + c_3\sinh\alpha L$$

Then $0 = 2c_3\sinh\alpha L$

since $\sinh \alpha L \neq 0$
Then $c_3 = 0$
Hence what is left with is the relation $0 = c_1 \sinh \alpha L$
A non-trivial solution $(c_1 \neq 0)$ only exists if $\sin \alpha L = 0 = \sin n\pi$
which means:

$$\alpha L = n\pi$$

And $\alpha = \dfrac{n\pi}{L}$
From equation (7)

$$\alpha^2 = \frac{\omega}{c} = \frac{n^2\pi^2}{L^2}$$

$$\omega = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}$$

Where ω is the natural frequency of the beam.
Substituting equation (15a) for equation (14) to obtain:

$$y(x,t) = \sum_{n=1}^{\infty}(A_n \cos wt + B_n \sin wt)\sin\frac{n\pi}{L}x \qquad (16)$$

Where A_n and B_n are constants which can be obtained from the initial conditions:
For initial displacement:

$$y(x,0) = f(x)$$

and initial velocity:

$$\frac{\partial y}{\partial t}(x,0) = g(x)$$

The constants A_n and B_n can be obtained as follows:
Substituting initial displacement for equation (16) yields:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \qquad (17)$$

Equation (17) is half range sine series[12]

Then

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}xdx \qquad (18)$$

Substituting initial velocity into equation (16) yields:

$$g(x) = \sum_{n=1}^{\infty} \omega B_n \sin\frac{n\pi}{L}x \qquad (19)$$

Then

$$B_n = \frac{2}{\omega L}\int_0^L g(x)\sin\frac{n\pi}{L}xdx \qquad (20)$$

1- Intermediate Concentrated Load:

If P is the concentrated load acting at distance x_1 from the left side of the beam as shown in Fig. (1).

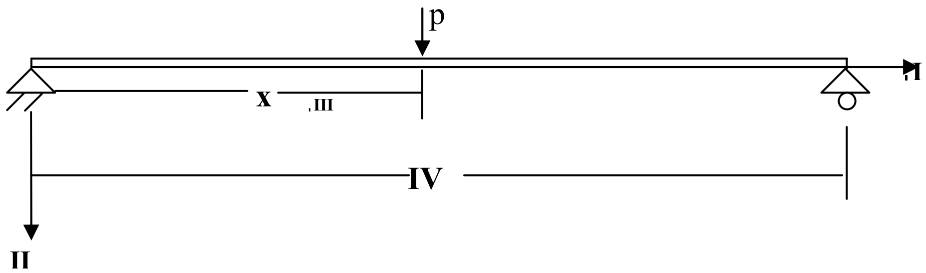


Fig .(1)

then the load function is:

$$w(x) = \begin{cases} 0 & 0 < \mathrm{x} < \mathrm{x}_1 - \frac{u}{2} \\ \lim\limits_{u \to 0} \frac{p}{u} & \mathrm{x}_1 - \frac{u}{2} < x < \mathrm{x}_1 + \frac{u}{2} \\ 0 & \mathrm{x}_1 + \frac{u}{2} < x < L \end{cases} \qquad (21)$$

The differential equation relating the deflection and the load is:

$$\frac{d^4 y}{dx^4} = \frac{w(x)}{EI} \qquad (22)$$

Representing the load w(x) by half range sine Fourier series [12]

$$w(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x \qquad (23)$$

where

$$C_n = \frac{2}{L} \int_0^L w(x) \sin \frac{n\pi}{L} x dx \qquad (24)$$

Substituting equation (21) for equation (24) then

$$C_n = \frac{2}{L} \lim_{u \to 0} \int_0^L \frac{p}{u} \sin \frac{n\pi}{L} x dx \qquad (25)$$

Integrating equation (25) gives:

$$\lim_{u \to 0} \frac{\sin u}{u} = 1$$

Then

$$C_n = \frac{2p}{L} \sin \frac{n\pi}{L} x_1 \qquad (26)$$

Substituting equation (26) for equation (23) then

$$w(x) = \sum_{n=1}^{\infty} \frac{2p}{L} \sin \frac{n\pi}{L} x_1 \sin \frac{n\pi}{L} \qquad (27)$$

In order to get the deflection due to the static load, it was assumed that the deflected shape represented by half range Fourier series:

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (28)$$

Which satisfies the boundary conditions of simply supported beams.

Substituting equations (28) and (27) for equation (22)

then

$$\sum \frac{n^4\pi^4}{L^4} b_n \sin\frac{n\pi}{L}x = \sum \frac{2p}{EIL}\sin\frac{n\pi}{L}x_1 \sin\frac{n\pi}{L}x \tag{29}$$

which yields:

$$b_n = \frac{2pL^3}{n^4\pi^4}\sin\frac{n\pi}{L}x_1 \tag{30}$$

Substituting equation (30) for equation (28) then

$$y(x) = \sum_{n=1}^{\infty}\frac{2pL^3}{n^4\pi^4}\sin\frac{n\pi}{L}x_1 \sin\frac{n\pi}{L}x \tag{31}$$

which represents the deflection equation due to the intermediate concentrated load.

If the load is suddenly removed the beam will vibrate freely and the initial displacement is the deflected shape at t=0, that means the equation (31) gives initial displacement in this case, then

$$y(x,0) = f(x) = \sum_{n=1}^{\infty}\frac{2PL^3}{n^4\pi^4}\sin\frac{n\pi}{L}x_1 \sin\frac{n\pi}{L}x \tag{32}$$

substitutes equation (32) for equation (17)

$$\sum_{n=1}^{\infty}\frac{2PL^3}{n^4\pi^4}\sin\frac{n\pi}{L}x_1 \sin\frac{n\pi}{L}x = \sum A_n \sin\frac{n\pi}{L}x \tag{33}$$

That means:

$$A_n = \sum_{n=1}^{\infty}\frac{2PL^3}{n^4\pi^4 EI}\sin\frac{n\pi}{L}x_1 \tag{34}$$

As the beam was at rest when the load was suddenly removed then the initial velocity is zero. That is:

$$g(x,0) = \frac{\partial y}{\partial t}(x,0) = 0 \tag{35}$$

If equation (35) is substituted for equation (16)

The value of B_n will be zero substitutes A_n and B_n into equation (16) then:

$$y(x,t)=\sum_{n=1}^{\infty}\frac{2PL^3}{n^4\pi^4EI}\sin\frac{n\pi}{L}x_1\sin\frac{n\pi}{L}x\cos\omega t \tag{36}$$

Which is the equation of free vibration for simply supported beam load by concentrated load at distance x_1 from the left end removed suddenly at time t=0.

If $x_1=\frac{L}{2}$

Then

$$y(x,t)=\sum_{n=1}^{\infty}\frac{2PL^3}{n^4\pi^4EI}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\omega t$$

Which is the same result that obtained by Timoshinko[11].

2- Partially Distributed Uniform Load:

Assuming that w/unit length the intensity of the uniform load acting at distance x_2 from the left side of the beam as shown in Fig. (2).

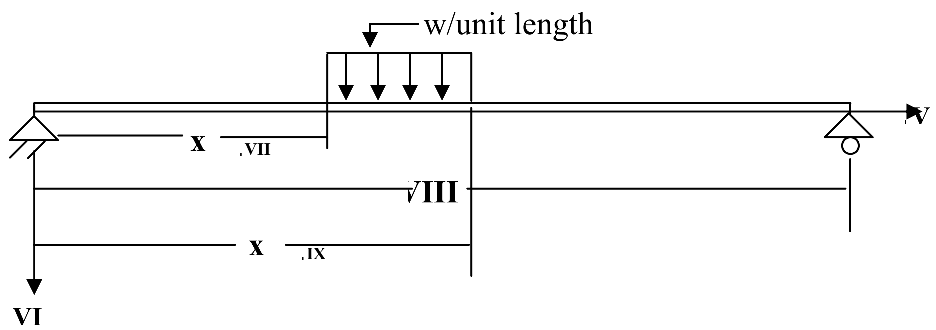


Fig.(2)

The load function is:

$$w(x)=\begin{cases}0 & 0<x<x_1\\ w & x_1<x<x_2\\ 0 & x_2<x<L\end{cases} \tag{37}$$

Representing the load w(x) by half range sine Fourier series [12]

$$w(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{L}x \tag{38}$$

Finding C_n from equation (24) after substituting equation (37) for equation (24) then:

$$C_n = \frac{2}{L}\int_0^L w \sin\frac{n\pi}{L} x dx \tag{39}$$

Integrating equation (39) to obtain:

$$C_n = \frac{2w}{n\pi}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right] \tag{40}$$

Then the solution of equation (38) becomes:

$$w(x) = \sum_{n=1}^{\infty}\frac{2w}{n\pi}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right]\sin\frac{n\pi}{L}x \tag{41}$$

Substitute equation (28) after rearranging the equation to obtain:

$$b_n = \frac{2wL^4}{n^5\pi^5 EI}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right] \tag{42}$$

Substituting equation (42) for equation (28) then

$$y(x) = \sum_{n=1}^{\infty}\frac{2wL^4}{n^5\pi^5 EI}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right]\sin\frac{n\pi}{L}x \tag{43}$$

Substituting equation (43) for equation (17) gives:

$$A_n = \frac{2wL^4}{n^5\pi^5 EI}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right] \tag{44}$$

If equation (35) is substituted into equation (16)

The value of B_n will be zero, substituting A_n and B_n for equation (16) then:

$$y(x,t) = \sum_{n=1}^{\infty}\frac{2wL^4}{n^5\pi^5 EI}\left[\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1\right]\sin\frac{n\pi}{L}x\cos\omega t \tag{45}$$

which is the equation of free vibration for simply supported beam loaded by uniform load acting at distance x_2 from the left side end and removed suddenly at time t=0.

If $x_1 = 0$ and $x_2 = L$
Then

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4wL^4}{n^5 \pi^5 EI} \sin \frac{n\pi}{L} x \cos \omega t$$

Which are the same results obtained by Timoshinko[11].

3- Intermediate Variable Load:

If (Q/unit length) the intensity of the uniform load acting at distance x_2 from the left side of the beam is as shown in Fig. (3).

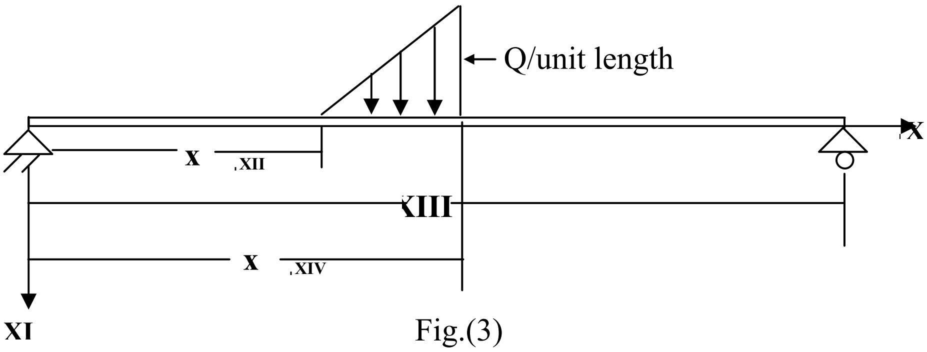


Fig.(3)

then the load function is:

$$w(x) = \begin{cases} 0 & 0 < x < x_1 \\ \dfrac{Q}{(x_2 - x_1)}(x - x_1) & x_1 < x < x_2 \\ 0 & x_2 < x < L \end{cases} \tag{46}$$

From equation (24) Then:

$$C_n = \frac{2}{L}\int_{x_2}^{x_1} \frac{Q}{(x_2 - x_1)}(x - x_1)\sin\frac{n\pi}{L}x dx \tag{47}$$

Solving equation (47) to obtain:

$$C_n = \frac{2QL}{(x_2 - x_1)n^2\pi^2}\begin{bmatrix} \sin\frac{n\pi}{L}x_2 - \sin\frac{n\pi}{L}x_1 \\ -\frac{n\pi(x - x_1)}{L}(\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1) \end{bmatrix} \tag{48}$$

then the solution of equation (23) becomes:

$$w(x) = \sum_{n=1}^{\infty}\frac{2QL}{(x_2 - x_1)n^2\pi^2}\begin{bmatrix} \sin\frac{n\pi}{L}x_2 - \sin\frac{n\pi}{L}x_1 \\ -\frac{n\pi(x - x_1)}{L}(\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1) \end{bmatrix}\sin\frac{n\pi}{L}x \tag{49}$$

From equation (28) after rearranging the equation to obtain:

$$b_n = \frac{2QL^5}{n^6\pi^6(x_2 - x_1)}\begin{bmatrix} \sin\frac{n\pi}{L}x_2 - \sin\frac{n\pi}{L}x_1 \\ -\frac{n\pi(x - x_1)}{L}(\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1) \end{bmatrix} \tag{50}$$

Substituting equation (50) for equation (28) yields:

$$y(x) = \sum_{n=1}^{\infty}\frac{2QL^5}{n^6\pi^6(x_2 - x_1)}\begin{bmatrix} \sin\frac{n\pi}{L}x_2 - \sin\frac{n\pi}{L}x_1 \\ -\frac{n\pi(x - x_1)}{L}(\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1) \end{bmatrix}\sin\frac{n\pi}{L}x \tag{51}$$

Substituting equation (51) for equation (17) gives:

$$A_n = \sum_{n=1}^{\infty}\frac{2QL^5}{n^6\pi^6(x_2 - x_1)EI}\begin{bmatrix} \sin\frac{n\pi}{L}x_2 - \sin\frac{n\pi}{L}x_1 \\ -\frac{n\pi(x - x_1)}{L}(\cos\frac{n\pi}{L}x_2 - \cos\frac{n\pi}{L}x_1) \end{bmatrix} \tag{52}$$

If equation (35) is substituted for equation (16),

the value of B_n will be zero, then substituting A_n and B_n for equation (16) to give:

$$y(x,t)=\sum_{n=1}^{\infty}\frac{2QL^5}{n^6\pi^6(x_2-x_1)EI}\begin{bmatrix}\sin\frac{n\pi}{L}x_2-\sin\frac{n\pi}{L}x_1- \\ \frac{n\pi(x-x_1)}{L}(\cos\frac{n\pi}{L}x_2-\cos\frac{n\pi}{L}x_1)\end{bmatrix}\sin\frac{n\pi}{L}x\cos\omega t$$

(53)

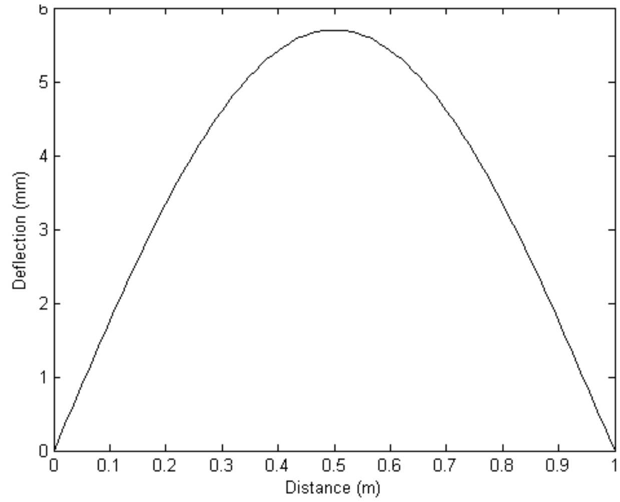
which is the equation of free vibration for simply supported beam loaded by a uniform load acting at distance x_2 from the left side end and removed suddenly at time t=0.

Numerical example:

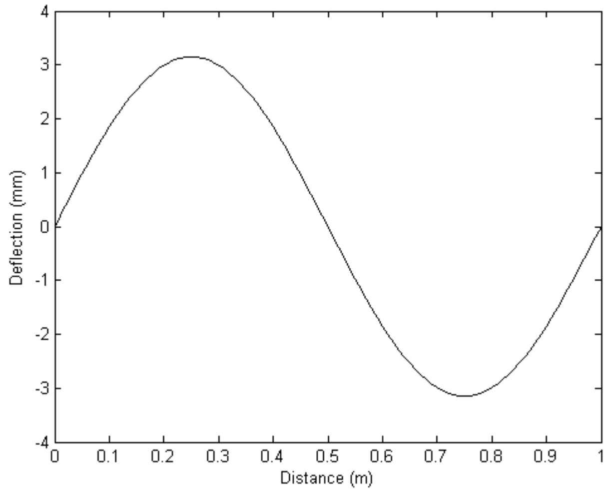
The following properties of a simply supported beam with uniformly distributed load will be considered to draw the mode shape of deflection: L=1m, E=200000 MPa, I=1.0666*10^{-3} m^4, w=300 N/m, m=7850kg/m^3 .
The relationship between deflection and the distance for these examples are shown in Fig.(4, 5) which are identical to that obtained by Timoskenko[11].

Conclusions:

The Fourier series method with separation of variables is suitable to be used for the solution of free vibration of beams. As the method is trigonometric (sine and cosine), then the deflection modes are of the same shape for different types of loads.
The solutions obtained are identical to those by Timoskenko[11] and Mustafa[10].



Mode 1



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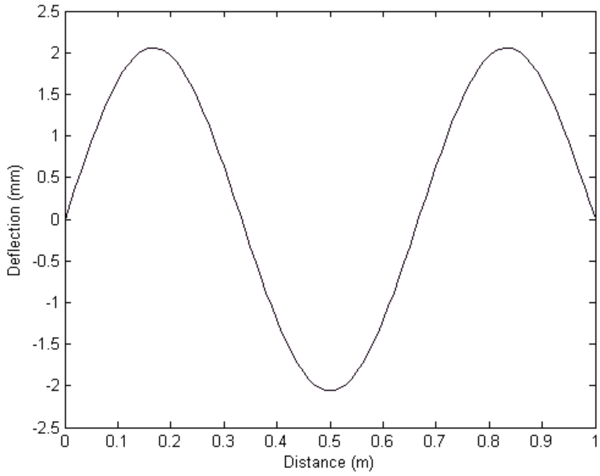
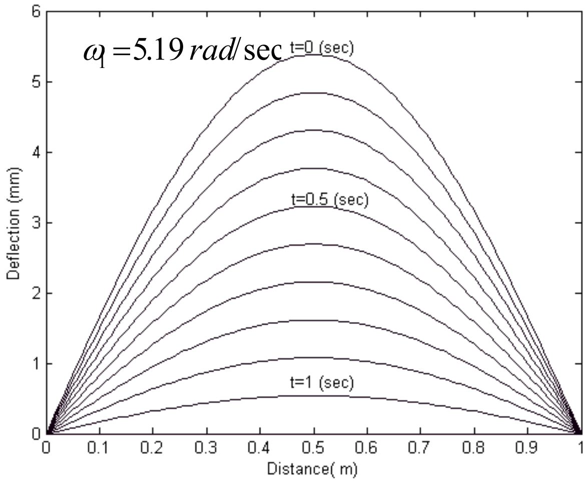


Mode 3

Fig.(4) Modes of deflection along the beam



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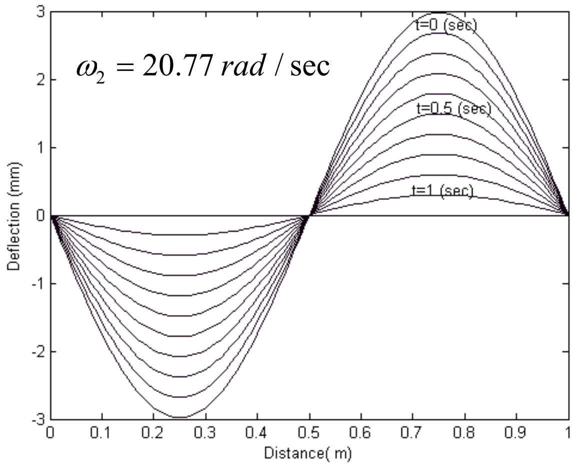

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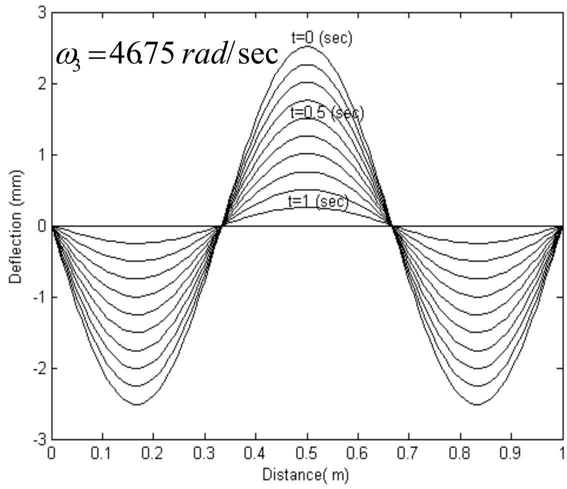

Mode 3

Fig .(5) Modes of deflection with time along the beam

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